JOURNAL OF COMPUTATIONAL PHYSICS 13, 158–160 (1973)

Note

Numerical Solution of Landau's Dispersion Equation

There is a continuing need for accurate values of Landau's pole, i.e., the root closest to the real axis of Landau's dispersion equation [1–3]:

$$K^2 + 1 + i\pi^{1/2} z w(z) = 0, \tag{1}$$

where $K = k\lambda_D$ (k is the wavenumber and λ_D is the Debye length); and $w(z)$ is the complex error function of long-standing importance in many branches of physics [4]. In plasma physics, $w(z)$ is generally known as the plasma dispersion function [5]. A survey of the plasma physics literature shows that: (i) although many researchers have solved (1) [6], no table of Landau's pole which is sufficiently complete and accurate seems to be available; (ii) there is a need in many plasma problems to compute the complex error function $w(z)$ with ease and accuracy, an obviously preferable procedure to table searches. The situation pointed out in (i) seems to this day to force the research worker interested in accurate values of Landau's pole to solve (1) again; while the observation in (ii) leads also to continued efforts by plasma physicists to develop new methods to compute $w(z)$.

The aim of this note is twofold: First, to give Landau's pole with five figure accuracy in the range of most interest $0.25 \leqslant K \leqslant 2.0$ with increments $\Delta K = 0.05$ (see Table I); simple linear interpolation from this table gives the frequency and damping rate for any K in the above range with four and three figure accuracy, respectively. Second, we wish to call attention to the important work of Gautschi [7, 8] on the computation of the complex error function $w(z)$. Gautschi's subroutine published in Algol [8] has been used in Fortran (a total of about 40 statements), and the values obtained for $w(z)$ were compared with the well-known tables of Faddeyeva and Terent'ev [9], and of Fried and Conte [5]. The efficiency and accuracy of Gautschi's subroutine proved excellent over the whole complex plane.

We now outline very briefly the method of solution of (1). It is known [1, 2] that the roots $z = x + iy$ of (1) are in the lower half plane (x gives the frequency and y the damping), $y < 0$. As Gautschi's subroutine evaluates $w(z)$ only in the first quadrant of the complex z-plane, we use the relation [7] $w(-z) = 2\exp(-z^2) - w(z)$ to transform (1) into an equivalent equation whose roots $z = x + iy$ satisfy $x > 0$, $y > 0$, and the corresponding Landau's root is $x_L = x$ and $y_L = -y$. Separating the real and imaginary parts in (1), we get a system of two nonlinear equations in x and y of the form:

$$f_1(x, y) = 0, \tag{2a}$$
$$K^2 + f_2(x, y) = 0. \tag{2b}$$

TABLE I

Numerical Solution of Landau's Dispersion Equation[a]

K	Real plasma frequency (ω/ω_0)	Damping coefficient $(-\gamma/\omega_0)$
0.25	1.1056	0.0021693
0.3	1.1598	0.012623
0.35	1.2209	0.034324
0.4	1.2850	0.066133
0.45	1.3502	0.10629
0.5	1.4156	0.15336
0.55	1.4809	0.20624
0.6	1.5457	0.26411
0.65	1.6100	0.32633
0.7	1.6739	0.39240
0.75	1.7371	0.46192
0.8	1.7999	0.53455
0.85	1.8621	0.61003
0.9	1.9239	0.68811
0.95	1.9851	0.76860
1.0	2.0459	0.85134
1.05	2.1062	0.93615
1.1	2.1662	1.0229
1.15	2.2257	1.1115
1.2	2.2848	1.2019
1.25	2.3436	1.2939
1.3	2.4020	1.3874
1.35	2.4600	1.4824
1.4	2.5178	1.5789
1.45	2.5752	1.6766
1.5	2.6323	1.7757
1.55	2.6892	1.8760
1.6	2.7457	1.9775
1.65	2.8020	2.0801
1.7	2.8580	2.1839
1.75	2.9138	2.2886
1.8	2.9693	2.3944
1.85	3.0246	2.5012
1.9	3.0797	2.6090
1.95	3.1345	2.7176
2.0	3.1891	2.8272
2.3088	3.5222	3.5522

[a] Landau's pole is given in units of the plasma frequency $\omega_0 = (4\pi ne^2/m)^{1/2}$.

where f_1 and f_2 are nonlinear functions of x and y involving the real and imaginary parts of $w(z)$. The fact that the parameter K appears in a simple form only in (2b) is exploited as follows. For a pair of guesses y_1 and y_2, the nonlinear equation (2a) is solved by Muller's method [10] for the corresponding roots x_1 and x_2; substitution of (x_1, y_1) and (x_2, y_2) in (2b) gives the corresponding values of K_1 and K_2. The approximate value y_3 corresponding to the desired value of K is now obtained by linear interpolation from the pairs (y_1, K_1) and (y_2, K_2). The y_3 thus obtained is now substituted into (2a) and this is again solved for x_3; the pair (x_3, y_3) gives now a new K_3, etc. Linear interpolation between the pairs (y_{i-1}, K_{i-1}) and (y_i, K_i) is pursued until

$$| K_i - K | < 10^{-6}. \qquad (3)$$

This accuracy criterion gives us five significant figure accuracy for the real and imaginary part of the root z. Except for $K = 0.25$ (see Table I), the number of linear interpolations required for convergence is about four, even when the initial guesses are quite inaccurate. For $K < 0.25$, convergence is difficult but Landau's pole can be obtained, if required, from asymptotic formulas [11].

References

1. L. Landau, *J. Phys. USSR* **10** (1946), 25.
2. J. D. Jackson, *J. Nucl. Energy C* **1** (1960), 171.
3. J. Canosa, IBM Scientific Center Rept. No. G320-3282, Palo Alto, California (1971).
4. B. H. Armstrong and R. W. Nichols, "Emission, Absorption and Transfer of Radiation in Heated Atmospheres," p. 218, Pergamon Press, New York, 1972.
5. B. D. Fried and S. D. Conte, "The Plasma Dispersion Function," Academic Press, New York, 1961.
6. See for example Ref. 2 above: S. P. Gary, *Phys. Fluids* **10** (1967), 570.
7. W. Gautschi, *SIAM J. Numer. Anal.* **7** (1970), 187.
8. W. Gautschi, *Comm. ACM* **12** (1969), 635.
9. V. N. Faddeyeva and N. M. Terent'ev, "Tables of Values of Function

$$w(z) = e^{-z^2}\left(1 + \frac{2i}{(\pi)^{1/2}} \int_0^z e^{t^2}\, dt\right)$$

for Complex Argument," Pergamon Press, New York, 1961.
10. D. E. Muller, *Math. Tables Aids Comp.* **10** (1956), 208. (Muller's method is programmed in a subroutine called RTMI available in IBM Corporation Report No. H20-0205-2, Technical Publications Dept., White Plains, New York.)
11. J. Canosa, *Phys. Fluids* **15** (1972), 1536.

Received: April 5, 1973

José Canosa

IBM Scientific Center
Palo Alto, California 94304